VALUE: Economics, Psychology, Life

Appendix Six: **Substitutability and Complexity**[1]

Goods come in categories, some narrowly defined, some broadly defined. We are looking for a measure of how *similar*, and by extension, of how substitutable, are the goods that belong to a single category.

Let J = {j}, j = 1, 2,....N, be an ordered list of all the relevant qualities or attributes common to the goods that belong to category J. Call {j}the defining *attribute-list* of J.

Let each unit or instance of the good belonging to a given category of goods for sale in a given market vicinity be uniquely numbered thus: g = 1, 2....G.

Let $g_j = 1$ when attribute j is *present* in unit good g, and let $g_j = 0$ when it is *not* present (or not sufficiently present) in g, but could be.[2]

Now consider any two unit goods, g and g' $(g \neq g')$, picked at random from all G unit goods. We are looking for a measure of how *similar* they are, and by extension, how *substitutable*-for-each-other they are likely to be perceived to be by a typical buyer. To do so, we construct the following sort of table, setting $M_j(g, g') = 1$ if $g_j = g'_j$ and $M_j(g, g') = 0$ otherwise:

j	g_j	g'_j	$M_j(g, g')$
1	0	0	1
2	0	1	0
3	1	1	1
4	1	0	0
5	0	0	1
6	0	0	1
7	1	1	1
.	.	.	.
.	.	.	.
.	.	.	.
N_J	0	1	0

The first column of this table is the category *attribute-list*, the second and third columns are individual good *checklists*, and the fourth column is a *matchlist*. A checklist is a binary "word" of length N that defines the nature of g to all present intents and purposes. A matchlist is a binary word of length N too, but one that contains information about the similarities and differences between two checklists.[3]

One straightforward measure of the *similarity* of g to g', SIM(g, g'), is given by the arithmetic operation of addition and averaging on the matchlist, to wit:

$$\mathrm{SIM}(g,\ g') = \frac{\sum_{j=1}^{N} \mathrm{M}_j(g, g')}{N}$$

Note that $0 \leq \mathrm{SIM}(g,\ g') \leq 1$. When $\mathrm{SIM}(g,\ g') = 1$, the two goods g and g' are maximally similar. Indeed, with respect to the N attributes that characterize the category, the two unit goods are effectively identical. We shall call them *twins*. When $0 < \mathrm{SIM}(g,\ g') < 1$, the two goods are just *similar*. And when, at the other extreme, $\mathrm{SIM}(g,\ g') = 0$, the two goods are maximally *dis*similar, i.e. as different as they can be from each other—within the constraint, of course, that they share the same attribute list in the first place. We shall call them *complements.*

As simple as it is, this set of distinctions is already interesting: for theoretically, twins necessarily have the same value to all buyers, similars might or might not have the same value, and complements cannot have the same value unless $\sum_{j=1}^{N} g_j = \sum_{j=1}^{N} g'_j = N/2$ (and even then it is unlikely).[4]

Now, if we randomly assigned "1"s and "0"s to their respective checklists, the chances that g and g' would be either twins or complements would be very slim. Below is the expression that gives us the probability that the similarity of two such "randomly composed" goods is equal to x.

$$\begin{aligned} \mathrm{p}[\mathrm{SIM}(g,\ g') = x] &= \binom{N}{Nx} \frac{1}{2^N}, \\ &= \frac{N!}{(Nx)!(N - Nx)!2^N} \end{aligned}$$

Studying the behavior of this expression teaches us three things. That

$\mathrm{p}[\mathrm{SIM}(\mathrm{g},\ \mathrm{g}') = 0] = \mathrm{p}[\mathrm{SIM}(\mathrm{g},\ \mathrm{g}') = 1] = \frac{1}{2^N}$,
p[SIM(g, g') = 0.5] is a maximum value for p[SIM(g, g') for all values of N, and
$\mathrm{p}[\mathrm{SIM}(\mathrm{g},\ \mathrm{g}') \geq 0.5] = 0.5$ for all values of N.

While it is always *possible* that two randomly-composed lists of N "1"s and "0"s are twins or complements, just as it is always *possible* to score "one-hundred percent" or zero by randomly filling out answers to multiple-choice exams (or always *possible* that two N-length strings of "heads" and "tails" selected at random from a sequence of $>N$ coin tosses could be identical or complementary), the most likely outcome, and especially as the number of questions (or coin tosses) is high, is that the exam-taker will score "fifty percent," and that the two coin-toss strings will have SIM(*string 1*, *string 2*) ≈ 0.5.

In real life, when g and g' belong to the same category of goods, we can be reasonably sure that any two instances of the good have $\mathrm{SIM}(g, g') > 0.5$. Indeed, this is what it means to have a useful category: not only that all the goods in that category share one attribute-list, which is a minimal requirement, but also that any pair of goods picked at random will be more similar to each other than they would be by chance. This higher-than-chance degree of similarity is what makes them substitutable for each other to a significant degree.[5]

Now consider the whole set of G goods belonging to a given category. There are $G(G-1)/2$ distinct pairwise similarity/dissimilarity comparisons that can be made between two unit goods. With all of these similarity computations tabulated, several statistics could be computed that would characterize the set of all G goods: totals, means, variances, and so forth. One suitable for our purposes is γ, the expected *substitutability* of the goods, defined thus:

$$\gamma \;=\; \mathrm{p}\left[\mathrm{SIM}(g, g') \geq 0.5\right]$$

When the goods are randomly composed, $\gamma = 0.5$. When $G \geq 2^N$, $\gamma = 0.5$ is the minimum value of γ. When the goods are identical to each other, "clones" or perfect commodities, then $\gamma = 1$. When the goods are substitutable to some extent, then γ lies between zero and one.

How does one arrive at a value of γ for non-randomly-composed goods? Only empricially, by an exhaustive tally of the number of times $\mathrm{SIM}(g, g') \geq 0.5$ is true relative to all possible cases that it could be true. Number every distinct pair $\{g, g'\} = 1, 2, \ldots. G(G-1)/2$. Then:

$$\mathrm{p}[\mathrm{SIM}(g, g') \geq 0.5] \;=\; \frac{1}{G(G-1)/2} \sum_{\{g,g'\}=1}^{G(G-1)/2} \mathrm{SIMcount}(g, g'),$$

where $\mathrm{SIMcount}(g, g') = 1$ if $\mathrm{SIM}(g,g') \geq 0.5$, and
$\mathrm{SIMcount}(g, g') = 0$ if $\mathrm{SIM}(g,g') < 0.5$.

In real life people do not do make exhaustive tallies of this sort, of course, but take a sampling of pairs and make estimates (sometimes quite far off) of whole-group γ and G.

How does *complexity* relate to this measure?

N is a direct measure of the potential complexity of every good in the category, at least with respect to its attributes.[6] There are 2^N ways to arrange N "0"s and "1"s in the checklist of a good, and so $C_{pot}(g) = \log_2 2^N = N$ bits, and since all goods in a category share the same attribute-list, N is the measure of the potential complexity of all the goods in the category. Since the probability that a pair of randomly-composed-goods-randomly-chosen will be twins or complements diminishes as $1/2^N$, as we saw earlier, it is reasonable to infer that—unless the goods were all identical (guaranteed!) in the first place—the greater their potential complexity the less likely it would be that it we would at random draw identical or complementary pairs from the whole group. This is our first indication that γ and N—substitutability and (potential) complexity—are inversely related, and it makes good sense: in general, the more complex an object, the greater is the number of criteria one can use to compare it to another object, and the more likely it is that any two objects, compared, will be thought to be different. The larger is the number of questions a teacher asks in a multiple choice exam, the finer can the teacher's grading discriminations be (at least at the level of "raw scores"). The more you look the more you see.

When many pairs of goods are evidently not "randomly composed" and have, rather, a degree of complexity that is less than their potential complexity, then γ, lying between 0.5 and 1, can be applied as a coefficient to C_{pot} to yield a measure that behaves like organization, R. The best closed-form expression that links γ to R through $C_{pot} = N$ is

$$\gamma \ = \ 0.5\left(1+\frac{R}{N}\right) \quad \text{or equivalently,} \quad R = (2\gamma - 1)N \ .$$

This gives us γ directly proportional to R for a given, constant value of N and inversely proportional to N for a given, constant value of R.

Everything I have said to this point about *goods* can be said about potential *buyers*. These individuals' similarity or dissimilarity to each other can be assessed by the same kind of computations as we used above, except that this time instead of tabulating the N_J salient attributes of the goods, we tabulate the N_K attributes of the buyers in as much as these characterize their varying yes-or-no *preferences* for the N_J salient attributes of the goods. With $N_J = N_K = N$, this yields

$$\beta \ = \ \mathrm{p}\big[\mathrm{SIM}(b,b') \geq 0.5\big], \quad B \geq 2^N$$

When potential buyers are "randomly composed" and they are quite dissimilar to each other, then $\beta = 0.5$. When potential buyers are all "of one mind," identical to each other in their tastes and sensitivities (at least *vis a vis* this category of goods), then $\beta = 1$. When $\beta = 1$, then as far as the goods (and their seller's) are concerned, one potential buyer is "as good as" another, and just as likely (or unlikely) to buy. Twin potential buyers are effecitively identical to each other, they like or dislike (or are indifferent, with a binary system indifference has to go with one or the other) the same attributes. Complementary potential buyers like the opposite attributes of a given good: what is desirable to one is undesirable (or of no matter) to the other.

We could do the same exercise again with *sellers*, as people with preferences, but recall that we decided—or rather *I* decided—that the personalities, environments, and conditions offered by sellers was sufficiently reflected in (how buyers thought of the) sellers' *goods*, that we could leave γ to bear the load of representing both variables.

Finally we might look at how buyers seek to *match* themselves to the available goods. Here our tabulation might look like this:

j, k	*b*	*g*	$M_{j,k}(b, g)$
1	0	0	1
2	0	1	0
3	1	1	1
4	1	0	0
5	0	0	1
6	0	0	1
7	1	1	1
.	.	.	.
.	.	.	.
.	.	.	.
N	0	1	0

where $b_j = 0$ means that buyer *b* disvalues (or is indifferent) to attribute *j*, and $b_j = 1$ means he values attribute *j*. Then $M_j(b, g) = 1$ if the buyer values an attribute the good has, or disvalues (or is indifferent) to an attribute a good does not have, and $M_j(b, g) = 0$ otherwise, i.e. if he disvalues (or is indifferent) to an attribute the good has, or values an attribute the good does not have. The quantity

$$\mathrm{SIM}(b, g) = \frac{1}{N} \sum_{j,k=1}^{N} M_{j,k}(b, g)$$

then becomes a measure of the degree of *match* between buyer b and good g. When b is not interested in g at all (except, perhaps, if she intends to re-sell g to people to whom SIM(b, g) > 0) SIM(b, g) = 0. When buyer b finds the good g "perfect," just what she wanted (including, if one of the judged attributes is price, the price), then SIM(b, g) = 1. For most goods likely to be sold in a marketplace SIM(b, g) > 0.5. And once again, the probability of a perfect match decreases with N, the total number of attributes displayed and sought, and therefore too with the potential complexity of both the good and the customer's desires. It is for this reason that the attributes of goods on the one hand, and of sets of desires on the other, must be *organized* in some way so as to decrease their complexity from $C_{\text{pot}} = N$ to something less than that. This increases the likelihood of buyers finding what they want and, conversely, of goods "finding" people who want them enough to buy them. This organizing activity is precisely what brokers and market makers provide, and why they deserve a share of the profits of sale.

Equation 8.3 uses the parameters β and γ to reflect the substitutability of buyers and goods among each other. The exponent μ_{match} reflects the probable degree of *match*:

$$\mu_{\text{match}} = \text{p}\left[\text{SIM}(b,g) \geq 0.5\right] \qquad G \text{ and } B \geq 2^N$$

so that the greater the degree of buyer-good match the more sensitive to competition is the seller's asking price P_s, and the less the less.[7] However, unlike β and γ whose limits are 0.5 and 1, in this case, because b and g are drawn from wholly different sets of checklists, it is possible for μ to vary from 0 to 1. (For example, imagine all buyers identical with $\{g_j\}$ = 0,0,0,0,1,0 and all goods identical with $\{g_j\}$ = 1,1,1,1,0,1. Here $\gamma = 1$, $\beta = 1$ but $\mu = 0$. Buyers preferences and good attributes are complementary.) When γ and β have magnitudes less than 1, it is possible to have μ_{match} = 1 or 0 as well as anywhere in between. However, in real life, as γ and β approach 0.5, and also as N increases, so the chances of high values for μ_{match} decrease, if only because of time and effort that might be required to match a highly variegated set of buyer preferences with a highly variegated set of goods attributes.

The μ that appears in Chapter Eight in Equation 8.3 is μ_{match} and more, because senstivity to competition depends on more than only the degree of match between set of buyer's preferences and the set of seller's goods. It depends too on the absolute value of G_{B_s} (the smaller is G_{B_s}, the bigger is μ), and on the value of good to the average competing buyer, $V_{b,\text{avg}}(g)$ (the greater is $V_{b,\text{avg}}(g)$ the greater is μ), and the terms of of payment, TP (the more flexible are the terms of payment offered by the seller, the more responsive are the buyers' offering prices to influence by other factors.) We can write:

$$\mu = F[\mu_{\text{match}}, G, V_{b,\text{avg}}(g), \text{TP}].$$

I have no definite form to offer for the function, F. I believe it is best determined empirical research, organized by category of good and market type. Certainly, $V_{b,\text{avg}}(g)$ and TP takes us into consideration of the buyer's perspective, which in Chapter Eight I analyze separately.

In Chapter Nine, where I discuss Gresham's Law using sets rather than ordered lists to describe a good's features, I use the set K = {k} to denote the subset of J that buyers actually value, i.e. for which $M_{j,k}(b, g) = 1$. It follows from the above discussion that the smaller is K relative to J, the lower is μ, all other things being equal.

Notes to Appendix Six

[1] This discussion owes much to Satosi Watanabe's treatment of "entropic measures of similarity and cohesion" in his *Knowing and Guessing: A Quantitative Study of Inference and Information* (New York, John Wiley and Sons, 1969), p. 409ff. Thanks also to Charles Friedman, Professor of Mathematics of the University of Texas at Austin. The mistakes and inadequacies that remain are my own.

[2] If the attribute-list $\{j\}$ is constructed to consist of only positively-valued attributes, then a *perfect* instance of a good in that category would have $g_j = 1$ for all j. This formalism introduces a crude consumer-buyer valuation measure where it does not belong in our model. I make note of the potential, however, as it comes up necessarily in the definition of α, below.

[3] We are assuming that all N attributes are themselves, *qua* attributes, equally valued (on average). But let us say that this is not true, and that one attribute, say $j = 4$, is more highly valued than the others. It is always possible to examine why this is so, and to replace it with k constituent attributes whose value is more equal to the others in the attribute list, which now expands to $N_{\text{new}} = N - 1 + k$.

The same can be done for judgments that are not easily cast into yes/no, 1/0, binary mode, as when one want to allow *degrees* of having a certain attribute (or of all attributes for that matter). But here again, let us say that one can distinguish m degrees of a good's having/not-having the attribute in question. One breaks this down into k binary judgments ($2^k = m$) and again $N_{\text{new}} = N - 1 + k$.

[4] Interestingly, while it is possible for a single category to consist of $G(G-1)/2$ twins, this occurring when all the goods are effectively the same (i.e. when they are "commodities"), it is not possible for a category to contain more than $G^2/4$ pairs of complements if one good can be partner to many, and $G/2$ *unique* pairs of complementary goods. These maxima obtain only if $G > 2^N$.

[5] Take any arbitrary collection of objects, objects not belonging to any useful category except that they all *exist* or *can-be-seen*. One can usually construct a category of $N \geq 1$ attributes, and a corresponding binary checklist, that groups some of these objects together and excludes others. Indeed, it is always possible to construct a category in such as way that its members are effectively identical, and nearly always possible construct a category that is not at all solipsistic and still has G identical members. But this gets harder to do as N gets larger.

At the other extreme, one can always construct a category whose defining checklist is so peculiar and long that only one object, and perhaps no *real* object, is a member of it. For this object; there could be no true replacement or substitute. It would be totally unique, "one of a kind" [although it might somewhere have an complementary twin (or two or three, identical to each other) to which its similarity is zero]. The only two things that can be totally unique would be things that had no attribute in common at all, not even "existing."

Many people like to think of *people* as being close to this unique, that every person is "in a class of their own." Connoisseurs like to think that every example of what they are connoisseurs *of* is this unique. But in truth, the only attribute, j, that *ensures* that SIM(g, g') $\neq 1$ for any g and g' is $j = \{x, y, z, t\}$ i.e. some precise spatiotemporal location. (Indeed, this is the function of space and time: to keep things apart so as to maintain the irreducible information content of the world.)

There can be no such thing as a category consisting of $G > 1$ goods that have *nothing in common*, not just because "categories" by definition, like "sets" in pure mathematics, depend on their being at least one common attribute or quality among all its members, but also because "having-nothing-in-common" is what they would have in common, and this is a logical paradox. A rough analogy: the Radical Individualists' Club. It can never meet, and its members cannot know they are members.

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[6] Of course, in its full reality as an object or experience, a good's true complexity is apt to be vastly higher than *N*—a not inconsequential fact about the economic world: recall our discussion of Gresham's Law in Chapter Eight)

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[7] How does one arrive at an actual value of μ? As with γ and β, by an exhaustive tally of the fraction of times SIM(b,g) ≥ 0.5 is the case to all *BG* possible cases that it could be, that is:

$$\text{p}[\text{SIM}(b,g) \geq 0.5] \;=\; \frac{1}{BG}\sum_{\{b,g\}=1}^{BG}\text{SIMcount}(b,g),$$

where SIMcount(b, g) = 1 if SIM(b, g) ≥ 0.5, and
SIMcount(b, g) = 0 if SIM(b, ,g) < 0.5.
